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Evaluation of a certain class of Eulerian integrals

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Abstract. In a study of the screening properties of a charged impurity located inside and near the surface of a metal subjected to a magnetic field, there arose an interesting class of Eulerian integrals, involving the Bessel function $J_\nu(z)$ or $J_0(z)$, which were expressed in closed forms by M L Glasser. Subsequently, L T Wille generalized and extended Glasser's work by giving closed-form expressions for a number of Eulerian integrals involving Meijer's G-function. Motivated by these recent works, we aim at evaluating a general class of Eulerian integrals involving Fox's H-function. Our main result (15) below is shown to provide a key formula from which one can deduce each of the aforementioned integrals and also numerous other potentially useful results not contained in the earlier works; we are led, in this way, to the corrected version of one of Wille's main integral formulae.

1. Introduction

Glasser (1984) gave closed-form expressions for an interesting class of Eulerian integrals, involving the Bessel function $J_\nu(z)$ or $J_0(z)$, which were encountered in a calculation of the screening properties of a charged impurity located inside and near the surface of a metal subjected to a magnetic field. In an attempt to provide generalizations of these Bessel-function integrals of Glasser (1984), Wille (1988) evaluated each of the following Eulerian integrals involving Meijer's G-function:

$$I_{x,y}^{m,n,p,q}(\alpha,\beta,a_p,b_q)=\int_0^1 t^x(1-t)^y G_{p,q}^{m,n}\left(\{\alpha^2(1-t)^{-1}+\beta^2t^{-1}\}^{-1}\,\middle|\,\begin{matrix}a_p\\ b_q\end{matrix}\right)\mathrm{d}t \tag{1}$$

$$J_{x,y}^{m,n,p,q}(\lambda,a_p,b_q)=\int_0^1 t^x(1-t)^y G_{p,q}^{m,n}\left(\lambda t(1-t)\,\middle|\,\begin{matrix}a_p\\ b_q\end{matrix}\right)\mathrm{d}t \tag{2}$$

and

$$K_{x,y}^{m,n,p,q}(\alpha,\beta,a_p,b_q)=\int_0^1 t^x(1-t)^y G_{p,q}^{m,n}\left(\frac{\{\alpha^2t+\beta^2(1-t)\}^2}{t(1-t)}\,\middle|\,\begin{matrix}a_p\\ b_q\end{matrix}\right)\mathrm{d}t \tag{3}$$

where, for convenience, a_p and b_q abbreviate the arrays of parameters

$$a_1,\ldots,a_n,\ldots,a_p \qquad \text{and} \qquad b_1,\ldots,b_m,\ldots,b_q \qquad (1\leqslant m\leqslant q;\,0\leqslant n\leqslant p)$$

respectively (Erdélyi *et al* 1953, Luke 1975, Mathai and Saxena 1973, Srivastava and Manocha 1984, and Srivastava *et al* 1982).

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The integral (2), as already pointed out by Wille (1988, p 601, equation (30)), is a special case of (1) when

$$\alpha^2=\beta^2=\lambda^{-1}.$$

In fact, much more general results than (2) can be found in the literature (Goyal 1969, Srivastava and Singh 1983). The object of the present sequel to Glasser (1984) and Wille (1988) is to evaluate a general class of Eulerian integrals which (upon suitable manoeuvres) would not only yield each of Wille's results (1), (2) and (3), but also provide closed-form expressions for numerous other potentially useful integrals not contained in the aforementioned works. Our investigation leads us naturally to the corrected version of Wille's integral formula associated with (3) above.

2. The general Eulerian integral and its evaluation

Meijer's G-function and its celebrated generalization, the H-function of Fox (1961), are usually defined in terms of Mellin–Barnes contour integrals involving quotients of Γ-functions. In fact, as already evidenced in the literature, such *single* Mellin–Barnes contour integrals are useful in the analytical solutions of various problems in nuclear and neutrino astrophysics (Mathai and Haubold 1988), and the Voigt functions $K(x, y)$ and $L(x, y)$ of astrophysical spectroscopy (and of the theory of neutron reactions) are expressible simply as *double* Mellin–Barnes contour integrals of this same class (Srivastava and Miller 1987); see also Buschman and Srivastava (1990) and the references cited there. Moreover, the H-function also includes, as its special cases, such mathematical functions as the Bessel–Wright function $J_\nu^\mu(z)$, the Fox–Wright function ${}_p\Psi_q$, the Mittag–Leffler functions E_α and $E_{\alpha,\beta}$, and the generalized parabolic cylindrical function (Srivastava and Manocha 1984 (p 50), and Srivastava *et al* 1982 (p 3)), which are not contained in Meijer's G-function occurring in Wille's integrals (1), (2) and (3). With these points in view, we address the problem of closed-form evaluation of the following general Eulerian integral involving Fox's H-function:

$$\mathscr{I}(z)\equiv\mathscr{I}^{\gamma,\delta,\lambda,\mu}_{\alpha,\beta,\rho,\sigma}\left[\begin{matrix}(a_j, A_j)_{1,p}; \\ (b_j, B_j)_{1,q};\end{matrix}\quad z;\xi,\eta\right]$$
$$=\int_\xi^\eta\frac{(t-\xi)^\lambda(\eta-t)^\mu}{\{f(t)\}^{\lambda+\mu+2}}H^{m,n}_{p,q}\left[z\{g(t)\}^\nu\,\middle|\,\begin{matrix}(a_j, A_j)_{1,p}\\ (b_j, B_j)_{1,q}\end{matrix}\right]\mathrm{d}t \tag{4}$$

where

$$f(t)=\eta-\xi+\rho(t-\xi)+\sigma(\eta-t) \tag{5}$$

$$g(t)=\frac{(t-\xi)^\gamma(\eta-t)^\delta\{f(t)\}^{1-\gamma-\delta}}{\beta(\eta-\xi)+(\beta\rho+\alpha-\beta)(t-\xi)+\beta\sigma(\eta-t)} \tag{6}$$

and $H^{m,n}_{p,q}[z|\ldots]$ denotes the familiar H-function of Fox (1961, p 408), defined by (Srivastava *et al* 1982 (chapter 2))

$$H^{m,n}_{p,q}\left[z\,\middle|\,\begin{matrix}(a_j, A_j)_{1,p}\\ (b_j, B_j)_{1,q}\end{matrix}\right]=\frac{1}{2\pi i}\int_{\mathscr{L}}\Theta(s)z^s\,\mathrm{d}s \tag{7}$$

where $\mathscr{L}$ is a suitable contour of the Mellin–Barnes type in the complex s-plane, and

$$\Theta(s)=\frac{\prod_{j=1}^m\Gamma(b_j-B_js)\prod_{j=1}^n\Gamma(1-a_j+A_js)}{\prod_{j=m+1}^q\Gamma(1-b_j+B_js)\prod_{j=n+1}^p\Gamma(a_j-A_js)}. \tag{8}$$

The symbolic form $(a_j, A_j)_{1,p}$, used in (4) and (7), and elsewhere in this paper, abbreviates the set of parameters

$$(a_1, A_1), \ldots, (a_p, A_p) \qquad (p \in \mathbb{N} = \{1, 2, 3, \ldots\})$$

the set being empty when $p = 0$. Also, the Mellin–Barnes contour integral representing the $\bar{H}$-function in (4) converges absolutely and defines an analytic function for

$$|\arg(z)| < \tfrac{1}{2}\pi\Omega \tag{9}$$

where

$$\Omega = \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j + \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p} A_j > 0 \tag{10}$$

A_j $(j = 1, \ldots, p)$ and B_j $(j = 1, \ldots, q)$ are positive real numbers.

We now proceed to evaluate the general Eulerian integral (4). Making use of the definition (7), we first find from (4) that

$$\mathscr{I}(z) = \int_\xi^\eta \frac{(t-\xi)^\lambda(\eta-t)^\mu}{\{f(t)\}^{\lambda+\mu+2}} \left(\frac{1}{2\pi \mathrm{i}} \int_{\mathscr{L}} \Theta(s) z^s \{g(t)\}^{\nu s} \,\mathrm{d}s\right) \mathrm{d}t \tag{11}$$

where $f(t)$, $g(t)$ and $\Theta(s)$ are given by (5), (6) and (8), respectively.

Assuming the inversion of the order of integration in (11) to be permissible by absolute (and uniform) convergence of the integrals involved above, we have

$$\mathscr{I}(z) = \frac{1}{2\pi \mathrm{i}} \int_{\mathscr{L}} \Theta(s) \left(\frac{z}{\beta^\nu}\right)^s \left(\int_\xi^\eta \frac{(t-\xi)^{\lambda+\nu\gamma s}(\eta-t)^{\mu+\nu\delta s}}{\{f(t)\}^{\lambda+\mu+\nu(\gamma+\delta)s+2}} \left\{1 - \frac{(\beta-\alpha)(t-\xi)}{\beta f(t)}\right\}^{-\nu s} \mathrm{d}t\right) \mathrm{d}s \tag{12}$$

where $f(t)$ and $\Theta(s)$ are given by (5) and (8), respectively.

If

$$|(\beta-\alpha)(t-\xi)| < |\beta f(t)| \qquad (t \in [\xi, \eta])$$

then use can be made of the binomial expansion, and we thus find from (12) that

$$\mathscr{I}(z) = \sum_{r=0}^{\infty} \frac{\{(\beta-\alpha)/\beta\}^r}{r!} \frac{1}{2\pi \mathrm{i}} \int_{\mathscr{L}} \Theta(s) \frac{\Gamma(\nu s + r)}{\Gamma(\nu s)} \left(\frac{z}{\beta^\nu}\right)^s$$
$$\times \left(\int_\xi^\eta \frac{(t-\xi)^{\lambda+r+\nu\gamma s}(\eta-t)^{\mu+\nu\delta s}}{\{f(t)\}^{\lambda+\mu+r+\nu(\gamma+\delta)s+2}} \mathrm{d}t\right) \mathrm{d}s \tag{13}$$

provided also that the order of summation and integration can be inverted.

The innermost integral in (13) can be evaluated by appealing to the following known extension of the Eulerian (beta-function) integral (Gradshteyn and Ryzhik 1980 (p 287, entry 3.198); see also Prudnikov *et al* 1983 (p 301, entry 2.2.6.1)):

$$\int_\xi^\eta \frac{(t-\xi)^{\alpha-1}(\eta-t)^{\beta-1}}{\{\eta-\xi+\lambda(t-\xi)+\mu(\eta-t)\}^{\alpha+\beta}} \mathrm{d}t = \frac{(1+\lambda)^{-\alpha}(1+\mu)^{-\beta}}{\eta-\xi} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{14}$$

$$\eta \neq \xi \qquad \mathrm{Re}(\alpha) > 0 \qquad \mathrm{Re}(\beta) > 0 \qquad \eta-\xi+\lambda(t-\xi)+\mu(\eta-t) \neq 0 \qquad (t \in [\xi, \eta])$$

and we finally obtain the desired integral formula:

$$\mathscr{I}(z) = (\eta-\xi)^{-1}(1+\rho)^{-\lambda-1}(1+\sigma)^{-\mu-1} \sum_{r=0}^{\infty} \frac{\{(\beta-\alpha)/\beta(1+\rho)\}^r}{r!}$$
$$\times \bar{H}^{m,n+3}_{p+3,q+2}\left[z\{\beta(1+\rho)^\gamma(1+\sigma)^\delta\}^{-\nu} \,\middle|\, \begin{matrix} (1-r, \nu), (-\lambda-r, \nu\gamma), (-\mu, \nu\delta), (a_j, A_j)_{1,p} \\ (b_j, B_j)_{1,q}, (1, \nu), (-\lambda-\mu-r-1, (\gamma+\delta)\nu) \end{matrix}\right] \tag{15}$$

which holds true when

(a) $\nu > 0$; $\gamma > 0$; $\delta > 0$; $\beta \neq 0$; $\eta \neq \xi$; $\rho, \sigma \neq -1$; and

$$\eta - \xi + \rho(t-\xi) + \sigma(\eta - t) \neq 0 \qquad (t \in [\xi, \eta])$$

(b) $\mathrm{Re}(\lambda + \nu\gamma(b_j/B_j)) > -1$ and $\mathrm{Re}(\mu + \nu\delta(b_j/B_j)) > -1$ $(j = 1, \dots, m)$

(c) m, n, p, q are integers constrained by

$$1 \leqslant m \leqslant q \qquad \text{and} \qquad 0 \leqslant n \leqslant p$$

(d) $|\arg(z)| < \frac{1}{2}\pi\Omega$, where Ω is given by (10);

(e) $|(\beta - \alpha)(t - \xi)| < |\beta\{\eta - \xi + \rho(t-\xi) + \sigma(\eta - t)\}|$ $(t \in [\xi, \eta])$; and

(f) the series on the right-hand side of (15) converges absolutely.

3. Applications

In this section we specifically show how the general integral formula (15) can be applied (and suitably manoeuvred) to derive various interesting (and potentially useful) results including those given by Wille (1988).

First of all, for $\rho = \sigma = 0$ and $z = (\eta - \xi)^{(\gamma+\delta-1)\nu}$, (15) readily yields

$$\mathscr{I}^{\gamma,\delta,\lambda,\mu}_{\alpha,\beta,0,0}\left[\begin{matrix}(a_j, A_j)_{1,p};\\ (b_j, B_j)_{1,q};\end{matrix}\quad (\eta-\xi)^{(\gamma+\delta-1)\nu}; \xi, \eta\right]$$
$$= (\eta - \xi)^{-1} \sum_{r=0}^{\infty} \frac{\{(\beta-\alpha)/\beta\}^r}{r!}$$
$$\times H^{m,n+3}_{p+3,q+2}\left[\left\{\frac{(\eta-\xi)^{\gamma+\delta-1}}{\beta}\right\}^{\nu} \left| \begin{matrix}(1-r, \nu), (-\lambda - r, \nu\gamma), (-\mu, \nu\delta), (a_j, A_j)_{1,p}\\ (b_j, B_j)_{1,q}, (1, \nu), (-\lambda-\mu-r-1, (\gamma+\delta)\nu)\end{matrix}\right.\right] \tag{16}$$

provided that the conditions easily obtainable from those of (15) are satisfied.

Setting $\beta = \alpha = 1/\kappa$ in (16), we obtain

$$\mathscr{I}^{\gamma,\delta,\lambda,\mu}_{1/\kappa,1/\kappa,0,0}\left[\begin{matrix}(a_j, A_j)_{1,p};\\ (b_j, B_j)_{1,q};\end{matrix}\quad (\eta-\xi)^{(\gamma+\delta-1)\nu}; \xi, \eta\right]$$
$$= (\eta-\xi)^{-1} H^{m,n+2}_{p+2,q+1}\left[\{\kappa(\eta-\xi)^{\gamma+\delta-1}\}^{\nu} \left| \begin{matrix}(-\lambda, \nu\gamma), (-\mu, \nu\delta), (a_j, A_j)_{1,p}\\ (b_j, B_j)_{1,q}, (-\lambda-\mu-1, (\gamma+\delta)\nu)\end{matrix}\right.\right] \tag{17}$$

which, in the further special case when

$$\xi = 0 \qquad \eta = 1 \qquad \mu = 0 \qquad \gamma = \delta = \nu = 1$$
$$A_j = 1 \qquad (j = 1, \dots, p)$$

and

$$B_j = 1 \qquad (j = 1, \dots, q)$$

would yield one of Wille's results (Wille 1988 (p 601, equation (29)) on using Legendre's duplication formula for the Γ-function.

Next we put

$$\gamma = \delta = 1 \qquad \lambda = \mu = -\tfrac{1}{2} \qquad \alpha \to \alpha^2 \qquad \text{and} \qquad \beta \to \beta^2$$

in the integral formula (16), and sum the resulting series by means of a known formula (Erdélyi *et al* 1953 (p 101, equation 2.8(6))); applying Legendre's duplication formula as well, we thus obtain the integral

$$\int_{\xi}^{\eta} \{(t-\xi)(\eta-t)\}^{-1/2} H_{p,q}^{m,n}\left[\left\{\frac{(t-\xi)(\eta-t)}{\alpha^2(t-\xi)+\beta^2(\eta-t)}\right\}^{\nu} \middle| \begin{matrix}(a_j, A_j)_{1,p}\\(b_j, B_j)_{1,q}\end{matrix}\right] \mathrm{d}t$$
$$=\sqrt{\pi}\, H_{p+1,q+1}^{m,n+1}\left[\left\{\frac{\eta-\xi}{(\alpha+\beta)^2}\right\}^{\nu} \middle| \begin{matrix}(\tfrac{1}{2}, \nu), (a_j, A_j)_{1,p}\\(b_j, B_j)_{1,q}, (0, \nu)\end{matrix}\right] \tag{18}$$

which, in the further special case when

$$\xi=0 \qquad \eta=1 \qquad \nu=1$$
$$A_j=1 \qquad (j=1, \ldots, p)$$

and

$$B_j=1 \qquad (j=1, \ldots, q)$$

immediately yields another result of Wille (1988, p 601, equation (22)).

If, in our integral formula (16), we set

$$\gamma=\delta=\tfrac{1}{2} \qquad \mu=-\lambda-2 \qquad \text{and} \qquad \nu\to 2\nu$$

sum the resulting binomial series, and apply Legendre's duplication formula once again, we shall obtain

$$\int_{\xi}^{\eta} (t-\xi)^{\lambda}(\eta-t)^{-\lambda-2}\, H_{p,q}^{m,n}\left[\frac{\{(t-\xi)(\eta-t)\}^{\nu}}{\{\alpha(t-\xi)+\beta(\eta-t)\}^{2\nu}} \middle| \begin{matrix}(a_j, A_j)_{1,p}\\(b_j, B_j)_{1,q}\end{matrix}\right] \mathrm{d}t$$
$$=2\sqrt{\pi}\,(\eta-\xi)^{-1}\left(\frac{\beta}{\alpha}\right)^{\lambda+1}$$
$$\times H_{p+2,q+2}^{m,n+2}\left[(4\alpha\beta)^{-\nu} \middle| \begin{matrix}(-\lambda, \nu), (\lambda+2, \nu), (a_j, A_j)_{1,p}\\(b_j, B_j)_{1,q}, (1, \nu), (\tfrac{1}{2}, \nu)\end{matrix}\right] \tag{19}$$

which holds true under the conditions readily obtainable from those stated with (15).

In terms of Meijer's G-function, an obvious special case of (19) when

$$\xi=0 \qquad \eta=1 \qquad \nu=1$$
$$A_j=1 \qquad (j=1, \ldots, p)$$

and

$$B_j=1 \qquad (j=1, \ldots, q)$$

can be deduced in the form

$$\int_0^1 t^{\lambda}(1-t)^{-\lambda-2} G_{p,q}^{m,n}\left(\frac{t(1-t)}{\{\alpha t+\beta(1-t)\}^2} \middle| \begin{matrix}a_p\\b_q\end{matrix}\right) \mathrm{d}t$$
$$=2\sqrt{\pi}\left(\frac{\beta}{\alpha}\right)^{\lambda+1} G_{p+2,q+2}^{m+2,n}\left(\frac{1}{4\alpha\beta} \middle| \begin{matrix}-\lambda, \lambda+2, a_p\\b_q, 1, \tfrac{1}{2}\end{matrix}\right) \tag{20}$$

or, equivalently,

$$\int_0^1 t^{\lambda}(1-t)^{-\lambda-2} G_{p,q}^{m,n}\left(\frac{\{\alpha t+\beta(1-t)\}^2}{t(1-t)} \middle| \begin{matrix}a_p\\b_q\end{matrix}\right) \mathrm{d}t$$
$$=2\sqrt{\pi}\left(\frac{\beta}{\alpha}\right)^{\lambda+1} G_{p+2,q+2}^{m+2,n}\left(4\alpha\beta \middle| \begin{matrix}a_p, 0, \tfrac{1}{2}\\\lambda+1, -\lambda-1, b_q\end{matrix}\right) \tag{21}$$

where we have made use of a familiar G-function identity (Erdélyi *et al* 1953 (p 209, equation 5.3.1(9))), together with some simple notational and parametric changes.

For $\lambda = x$, $\alpha \to \alpha^2$ and $\beta \to \beta^2$, this last integral formula (21) would provide the *corrected* version of Wille's result (1988, p 602, equation (33)) for the integral

$$K^{m,n,p,q}_{x,-x-2}$$

defined by (3) above.

Several other potentially useful integral formulae can be deduced, in this manner, from our key result (15) and its consequences considered here.

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